Optimal Mechanism for Selling Two Goods with Uniformly Distributed Valuations

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- Based on the value of $\hat{z}$, the auctioneer decides the allocation probability $q : D \to [0, 1]^2$, and the payment from the buyer $t : D \to \mathbb{R}$.
- Consider a quasi-linear mechanism, where the utility function of the buyer is given by $u(z, \hat{z}) = z \cdot q(\hat{z}) - t(\hat{z})$.

Optimal Auctions

- **Auctioneer's objective**: Maximize $\mathbb{E}_z t(z)$, subject to IC and IR constraints.
- **IC**: $u(z) \geq u(z, \hat{z}) \, \forall z, \hat{z} \in D$.
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- Define $\phi(z) = z - \frac{1-F(z)}{f(z)}$. Assume that ϕ is increasing. Then the optimal allocation is given by

$$q(z) = \begin{cases} 1 & \text{if } \phi(z) \geq 0 \\ 0 & \text{if } \phi(z) \leq 0. \end{cases}$$

- Myerson has also provided the solution for the single-item n-buyer setting.

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Two-item optimal auctions

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- Cases such as $f = Unif[0, b_1] \times [0, b_2]$, $f = \exp(\lambda_1) \times \exp(\lambda_2)$, and $f = \text{Beta} \ \times \text{Beta}$ belong to this category.

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- The optimal solution for all these cases was given by Daskalakis et al.[2]

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- The optimal solution for all these cases was given by Daskalakis et al.[2]
- In this talk, we shall derive the formulation of the optimization problem as done by Daskalakis et al., discuss their solutions when $f = Unif[0, b_1] \times [0, b_2]$, and then the work that we have done to find the optimal solution for the case when $f = Unif[c_1, c_1 + b_1] \times [c_2, c_2 + b_2]$.

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Primal problem

- Recall that the auctioneer's objective was to maximize $\mathbb{E}_z t(z)$ w.r.t. IC and IR constraints.
- Rochet's theorem provides a necessary and sufficient condition for a mechanism to be IC and IR.

Theorem

A quasi-linear mechanism satisfies IC and IR, iff $u(\cdot)$ is convex, $\nabla u(z) = q(z)$ a.e. $z \in D$, and $u(z) \geq 0 \, \forall z \in D$.[3]

- This theorem helps us formulate the optimization problem completely in terms of u.

$$\max_u \int_D (z \cdot \nabla u(z) - u(z)) f(z) \, dz$$
$$\text{s.t. } \{u \text{ convex}, \nabla u(z) \in [0,1]^2 \text{ a.e. } z, \text{ and } u(z) \geq 0 \, \forall z.\}$$

[3] J.C.Rochet, "The Taxation Principle and the Multi-time Hamilton Jacobi Equations", Journal of Mathematical Economics 14, 2 (April 1985), 113â128.

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- $\nabla u(z) \in [0,1]^2 \Leftrightarrow u(x) - u(y) \leq c(x,y) \, \forall x, y \in D^2$, where, $c(x,y) = (x_1 - y_1)_+ + (x_2 - y_2)_+$.

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- $u(z) \geq 0 \Leftrightarrow u(0,0) \geq 0$, since $u(0,0) \geq 0$ combined with $\nabla u \geq 0$ implies $u(z) \geq 0$.
- We further consider $u(0,0) = 0$, since fixing $u(0,0) > 0$ only reduces the objective function.

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- We further consider $u(0,0) = 0$, since fixing $u(0,0) > 0$ only reduces the objective function.
- The optimization problem now becomes

$$\max_u \int_D (z \cdot \nabla u(z) - u(z)) f(z) \, dz$$

$$\text{s.t. } \{u \text{ convex}, \; u(x) - u(y) \leq c(x,y) \, \forall (x,y), \text{ and } u(0,0) = 0.\}$$

Primal problem (contd...)

- Using integration by parts, the objective function can be written as,

$$\int_D (z \cdot \nabla u(z) - u(z)) f(z)\, dz = \int_D u\, d(\mu + \mu_s)$$

- $\mu(z) := -z \cdot \nabla f(z) - 3f(z)$, $\mu_s(z) := f(z)(z \cdot n)\mathbf{1}(z \in \partial D)$.

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- We have $\int_D d(\mu + \mu_s) = -1$.
- To make this 0, we add a point measure μ_p of 1 at $(0, 0)$.
- Defining $\bar{\mu} = \mu + \mu_s + \mu_p$, we have the objective function to be $\int_D u\, d\bar{\mu}$. Observe that defining μ_p causes no harm to the objective function since $u(0, 0) = 0$.

Primal and the dual

The Primal problem:

$$\max_u \int_D u \, d\bar{\mu}$$

$$\text{s.t. } \{u \text{ convex }, \; u(x) - u(y) \le c(x,y) \, \forall (x,y), \; u(0,0) = 0.\}$$

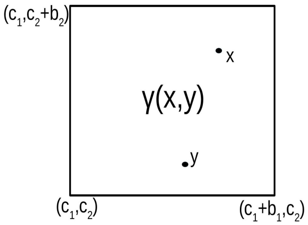
The Dual problem:

$$\inf_\gamma \int_{D \times D} c(x,y) \, d\gamma(x,y)$$

$$\text{s.t. } \{\gamma(\cdot, D) = \gamma_1(\cdot), \; \gamma(D, \cdot) = \gamma_2(\cdot), \; \text{and } \gamma_1 - \gamma_2 \succeq_{cvx} \bar{\mu}.\}$$

where we say the measure α *convex dominates* measure β ($\alpha \succeq_{cvx} \beta$) if for every convex and increasing function, we have $\int_D f \, d\alpha \ge \int_D f \, d\beta$.

The Optimal Transport problem



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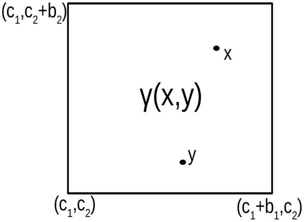

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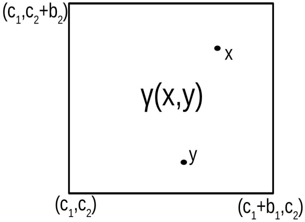

- The dual problem is a version of *optimal transport* problem.
- $c(x, y) \rightarrow$ Cost of transporting unit mass from x to y.
- $\gamma(x, y) \rightarrow$ The differential mass transported from x to y.
- We need to find a way to minimize the cost of transportation subject to the constraint that $\gamma_1 - \gamma_2 \succeq_{cvx} \bar{\mu}$.

The solution by Daskalakis et al.

- The problem of optimal transport was solved by Daskalakis et al. for f_1 and f_2 that give rise to a *well-formed* canonical partition of the support set D.

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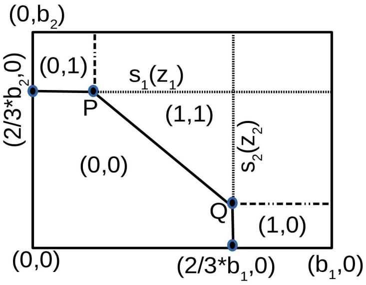

where the line joining P and Q is $z_1 + z_2 = \frac{2b_1+2b_2-\sqrt{2b_1 b_2}}{3}$.

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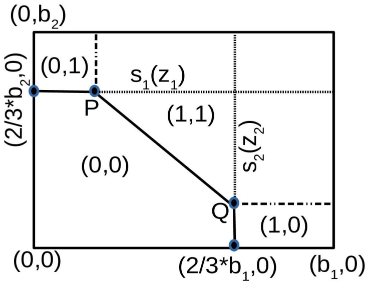

where the line joining P and Q is $z_1 + z_2 = \frac{2b_1+2b_2-\sqrt{2b_1 b_2}}{3}$.

- The optimal γ that they provide is such that $\gamma_1 - \gamma_2 = \bar{\mu}$.

Uniform $[c_1, c_1 + b_1] \times [c_2, c_2 + b_2]$

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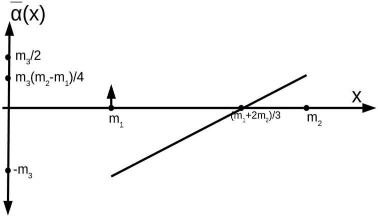

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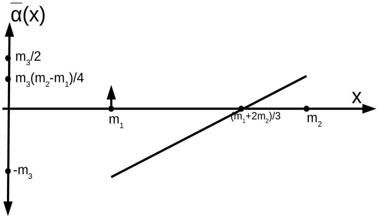

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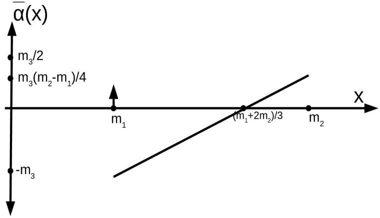
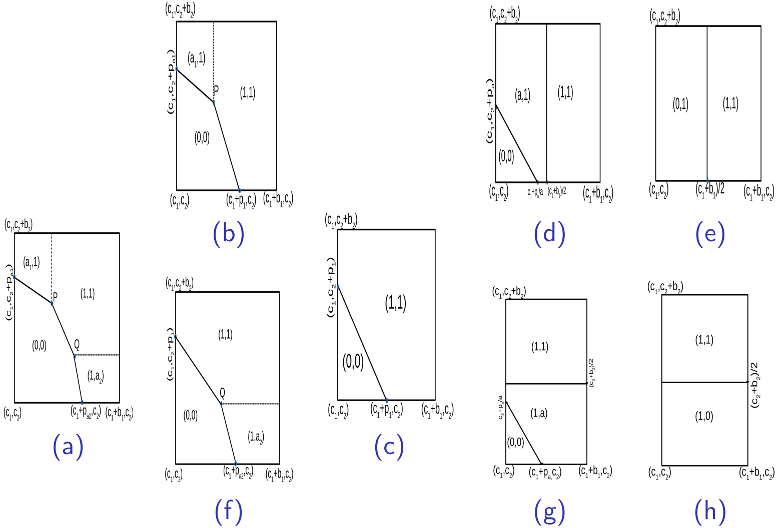


- One can prove that $\bar{\alpha} \succeq_{cvx} 0$ for any $m_2 \geq m_1 \geq 0$, $m_3 \geq 0$.
- We thus find a solution which does not relax the convexity constraint.

The general structure



Conjecture

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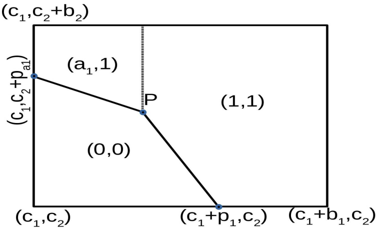
Consider $z \sim Unif[c_1, c_1 + b_1] \times [c_2, c_2 + b_2]$*. The structure of the optimal solution for any* $(c_1, c_2, b_1, b_2) \in \mathbb{R}_+^4$ *is one among the eight structures (a)-(h).*

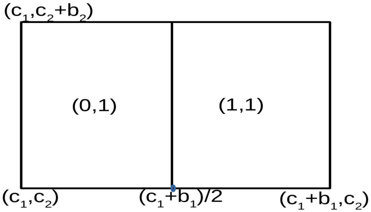
In other words, defining E_x to be the set of all (c_1, c_2, b_1, b_2) such that the optimal solution has the structure "x", "x" taking any alphabet from (a) to (h), we conjecture that $\cup_{x=a}^{h} E_x = \mathbb{R}_+^4$.

Structures when $c_1 = 0$

Theorem

Consider $z \sim Unif[0, b_1] \times [c_2, c_2 + b_2]$. Then the optimal solution has one of the following structures when $\frac{b_1}{b_2} \geq 2$:





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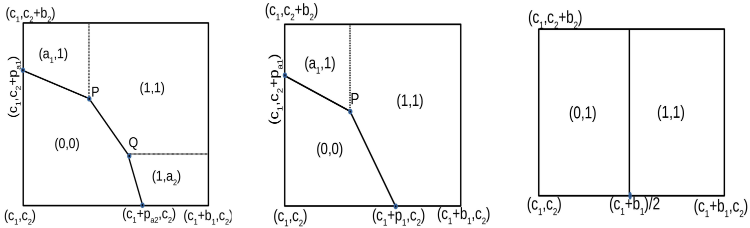
The optimal solution has one of the following structures when $\frac{b_1}{b_2} \in [0.6541, 2]$:



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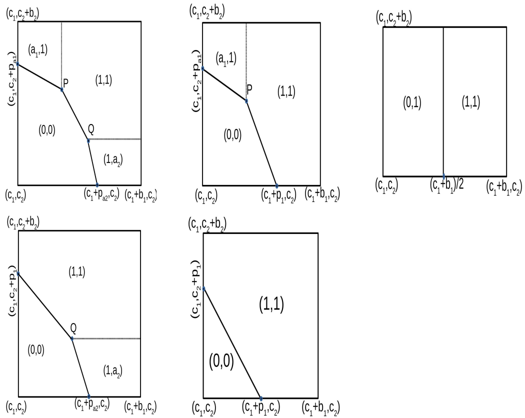

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- Identifying that the solution was found by relaxing the convexity constraint, we found a "shuffle measure" $\bar{\alpha}$, and tried to construct a γ that has its convexity constraint tight.

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- Identifying that the solution was found by relaxing the convexity constraint, we found a "shuffle measure" $\bar{\alpha}$, and tried to construct a γ that has its convexity constraint tight.
- We conjecture that the optimal solution satisfies one of the eight structures given by the shuffle measure.
- The optimal solution was proved to have one of those structures when $c_1 = 0$.

Future Work

- Can the conjecture be proved? Or can we find a counter-example of a (c_1, c_2, b_1, b_2) whose structure of optimal solution is different from the eight?

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- How do we find the optimal solution when the distribution of valuations is not uniform? Can we derive a general solution for finding the solution for any distribution?
- What happens when the number of buyers is more than 1?